The Triumvirate of an Adaptive Criteria Weighting Methodology, Isomorphic Comparator Similarity Measure, and Apropos High Dimensional Data Cluster Validation Index Measures for the Ascertainment of Bespoke Dynamic Fuzzy Lists

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Abstract—**Dynamic Real World Systems (RWS) must contend with the temporal dimension relating to the membership function and the associated RWS phenomenon of uncertainty. This problem is particularly interesting because it currently constitutes an impediment to robust Information Fusion (IF). There is a need for a more capable mechanism, pertaining to the aforementioned, for IF, forecasting, and decision-making, and central to this is the Dynamic Fuzzy List (DFL). Historically, a variety of foundational theories and works have been considered, but ascertaining the extent of affinity, within an acceptable performance time frame, has been an ongoing challenge within High Dimensional Data (HDD). This paper explored a prospective HDD-centric triumvirate amalgam consisting of: (1) an Adaptive Criteria Weighting Methodology (ACWM) to derive entropy weights, (2) an Isomorphic Comparator Similarity Measure (ICSM) to gauge similitude, via the discernment of Very Small/Non-Obvious (VSNO) clusters in HDD, and (3) apropos HDD-centric Cluster Validity Index (CVI) Measures (HCM) that are data uncertainty-centric to address the discussed need. The main output of this paper is that of an ACWM-ICSM-HCM triumvirate approach, as a contribution to the DFL challenge — and the overarching complex problem of risk-prediction-decision in advanced Artificial Intelligence (AI)/Machine Learning (ML) systems — by addressing a thematic that is crucial for IF in dynamic RWS and well warrants further investigation.**

Keywords-Computational Intelligence Strategies; Future Computational Technologies; Artificial Intelligence; High Dimensional Data; Isomorphic Comparator; Information Fusion; Dynamic Systems; Dynamic Fuzzy List.

I. INTRODUCTION

There is an emergent type of dynamic fuzzy set that is being looked to by certain organizations. It is, in essence, a Dynamic Fuzzy List (DFL), wherein membership is not necessarily fixed or certain and may fluctuate temporally. Hence, it has characteristics of Zadeh's Type-2 Fuzzy Set (T2FS), wherein membership is both uncertain and fluid, as well as that of Yang & Hinde's Rough (R)-Fuzzy Set (RFS), wherein the membership value of the prototypical constituent element of the upper bounds "has an affinity, but not necessarily absolute inclusion" [1]; it also has characteristics drawn from other foundational facets, which will be discussed in Section III, but suffice it to say, deriving such a DFL is a non-trivial matter. For example, candidacy for the DFL can be predicated upon a variety of considerations, such as criteria weighting, a particular snapshot in time, and/or the discernment of related entities (which could affect the membership candidacy). Each of these constitutes a considerable challenge. For example, first, Keshavarz-Ghorabaee notes that the determination of criteria weights "is one of the most critical and complicated processes" [2]. Second, as the criteria for membership may fluctuate with time and the desire for membership is uncertain and fluid, the paradigm at a particular instance in time (with its myriad of parameters) should be compared against known isomorphic paradigms for Quality Assurance/Quality Control (QA/QC) purposes. An Isomorphic Comparator Similarity Measure (ICSM) is needed to gauge the similarity of the paradigm with prior instances. Yet, the resultant of the ICSM may not even be needed if a Lower Ambiguity, Higher Uncertainty (LAHU) and Higher Ambiguity, Lower Uncertainty (HALU) Module (LHM) determines that it will not seek further resultants. More on this will be presented in Section III. Third, the recognition of affinity (e.g., temporally fluctuating dotted-line relationships) is particularly challenging, as even the seemingly simplistic task of mapping the entities associated with a large company can be complex, as it is difficult to ascertain the extent of affinity (or even control/subordination) one entity might have with/over/under another (e.g., as pertinent case studies, the described paradigm is reflected in the notions of "keiretsu," "chaebol," "qiyejituan," etc.). In numerous cases, certain unknown entities have been the "point of the spear" for technological initiatives and/or operational programs/functional roles. Interesting historical case studies include Atari, Inc.'s obfuscated subsidiary Kee Games (thought to be a competitor) [3] and Aisin Seiki (an obscured single-source supplier of a brake-related part for the Toyota Group, which includes Toyota Motor), which experienced a fire at one of its plants that could have impacted/stopped operations at Toyota for weeks; however, the full might of the Toyota Group keiretsu was illuminated as subsidiaries, suppliers/contractors, and affililates engaged in helping produce the brake-related part (e.g., a sewing machine manufacturer, within the keiretsu, shifted its production line to start making the brake-related part) [4]. Accordingly, the identification of entities with an affinity (e.g., under a keiretsu) is challenging to say the least.

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Restated in relationship/network science terms, these difficult to discern, but pertinent sparse clusters, particularly for the study space of this paper, can be deemed to be Very Small/Non-Obvious (VSNO); furthermore, these VSNO need to be detected within High Dimensional Data (HDD), wherein the number of features, f, are approximately equal to or greater than the number of observations, o. This HDD VSNO Discernment challenge is referred to as HVD.

Discernment engines have been among the most coveted technological goals within ecosystems that span multi-domains. Computer vision, communications, and other arenas have long sought Artificial Intelligence (AI)-centric Software-Defined approaches for high-resolution, multi-resolution image and signal analyses, among other purposes [5]. Yet, noise and vagaries, among other factors, have plagued a number of efforts. Recently, AI implementers have increasingly explored DFL as a potential means to capture and embody the vagueness of a paradigm [6]; this is crucial for the ICSM. Traditionally, high-precision Numerical Methods (NM) were utilized to contend with emulating/modeling complex Real World Systems (RWS); however, there has existed a gap in contending with and capturing the RWS paradigm/phenomenon of vagueness/fuzziness; the available/involved NM as well as the associated computational cost to bridge this gap have struggled with being robust/feasible enough to satisfy the challenge. The literature delineates various surrogate/proxy approaches that endeavor to enhance the performance for the involved AI//ML-based RWS (e.g., by reducing the training time) [7]. Other approaches have involved accelerations via a SOT while yet others have focused upon preserving the multi-scale structure of the involved matrices via the involved SOT [8]. Regardless of the approach, CVI affirmation is a key step in the overall process and is instrumental in the choice(s) for the involved clustering algorithm(s)/heuristic(s); these decisions directly impact the computational performance for the involved RWS-at-large.

The aspects discussed within this paper are presented in Table I (with utilized acronyms), via five parts: (1) the overarching objectives (e.g., targets, actions), (2) the functional requirements, (3) the constraints (e.g., functional, selection bias, temporal), (4) some specific boundaries, and (5) the requisite components (e.g., the triumvirate constituents, which each constitute a separate system). In this way, by Section IV (Discussion, Conclusion, & Future Work), it can be evaluated whether the proposed triumvirate approach suffices in addressing the overarching objectives.

TABLE I. CONSIDERED ASPECTS OF THE TRIUMVIRATE AMALGAM (WITH UTILIZED ACRONYMS)

I. Overarching Objectives (e.g., targets, actions) *Decision Support & Decision-Making*
• Optimal Decision Engineering Pathway (DEP) amidst ➢Uncompressed Decision Cycles (UDC) ➢Compressed Decision Cycles (CDC) • Multi-Criteria Decision Making (MCDM) with constituent ➢Multi-Attribute Decision Making (MADM) ➢Multi-Objective Decision Making (MODM) which should be considered via Mathematical Programming Methods (MPM), Artificial Intelligence (AI)/Machine Learning (ML), and Integrated Approaches (IA) • Quality Assurance/Quality Control (QA/QC)
II. Functional Requirements *Aspects Needed*
• Dynamic Fuzzy List (DFL) represented by Numerical Methods (NM) for Real World Systems (RWS) • HDD VSNO Discernment (HVD) in High Dimensional Data (HDD) for Very Small/Non-Obvious (VSNO) wherein the requirements above are dictated by a • LAHU/HALU Module (LHM), which is comprised of ➢Lower Ambiguity, Higher Uncertainty (LAHU) ➢Higher Ambiguity, Lower Uncertainty (HALU) considerations
III. Constraints (e.g., functional, bias, temporal) *Implementation Considerations*
• Relationship Membership Stream (RMS) ➢Probability [& Statistics] Systems Theory (PST) ➢Fuzzy Systems Theory (FST) ▻Type-2 Fuzzy Set (T2FS), as contrasted to Type-1 Fuzzy Set (T1FS) -Footprint of Uncertainty (FOU) ▻Spherical Fuzzy Sets (SFS) ▻Three-Way Soft Clustering (TWSC) ➢Rough Set (RS) ➢Rough (R)-Fuzzy Set (RFS) ➢Grey Systems Theory (GST) • AI/ML Metaheuristic Limitations ➢Particle Swarm Optimization (PSO) ▻Constriction Factor (CF) ▻Robust Convex Relaxation (RCR) ▻Long Short-Term Memory (LSTM) ▻Deep Convolutional Neural Network (DCNN) ▻CF-PSO-RCR-LSTM-DCNN (CPRLD) ➢Sequence of Transformations (SOT) ▻Nonnegative Matrix Factorization (NMF) ▻Gaussian Composite Model (GCM) ▻Multiresolution Matrix Factorization (MMF) ▻Corresponding WT (CORWT) ▻Enhanced CORWT (ECORWT) ▻Wavelet Transform (WT) which include -Discrete WT (DWT) -Stationary Wavelet Transform (SWT) -Continuous Wavelet Transform (CWT), whose implementation can include CWT PyWavelet Schema (CPS) • Explainable AI (XAI) ➢Criteria Weighting Systems (CWS), which might utilize ▻Subjective Methods (SMs) ▻Objective Methods (OMs) which might include MADM/MODM SM/OM (MMSO), such as ▻Point Allocation (PA) ▻Analytic Hierarchy Process (AHP) ▻CRiteria Importance through Intercriteria Correlation (CRITIC) ▻Technique of Order Preference by Similarity to an Ideal Solution (TOPSIS) ▻Multi-Objective Optimization by a Ratio Analysis plus the Full Multiplicative Form (MULTIMOORA) while other HDD-oriented sub-space approaches include ▻Clustering in QUEst (CLIQUE) ▻Merging Adaptive Finite Intervals And (MAFIA) ➢Adaptive CWS (ACWS), which underpins the ACWM • Overall TFCP, which is comprised of ➢Transparency (TY)

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➢Flexibility (F) ➢Consistency (C) ➢Performance (P)
IV. Specific Boundaries
• Cluster Validity Index (CVI), which can be grouped as ➢External Measures (EMs), such as ↝F-Measure (FM) ↝Normalized Mutual Information (NMI) ➢Internal Measures (IMs), such as ↝Calinski-Harabasz (CH) ↝Davies-Boulding (DB) ↝Ball-Hall (BH) ↝Pakhira-Bandyopadhyay-Maulik (PBM) ↝Trace(W) (TW) ↝Point-Biserial (PB) ➢Relative Measures (RMs), which can be construed to be IMs, such as ↝Dunn-Index (DI) ↝Maulik-Bandyopadhyay (MB) ↝McClain-Rao (MR) These can also be grouped as ↝Difference-like Criteria (DLC) ↝Optimization-like Criteria (OLC)
V. Requisite Components (e.g., the triumvirate constituents, which each constitute a separate system) *Triumvirate Constituents*
• Adaptive Criteria Weighting Methodology (ACWM) • Isomorphic Comparator Similarity Measure (ICSM) • HDD CVI Measures (HCM)

In brief, this paper delineates a proposed triumvirate approach for the ascertainment of certain desired DFLs. The paper is structured as follows. Section I provided a backdrop and introduces the problem space of DFLs. Section II provides background information by way of describing the operating environment as well as the state of the challenge for contending with certain selection bias in Criteria Weighting Systems (CWS), the discernment of VSNO in HDD, and the selection of data uncertainty-centric HDD CVI Measures (HCM). Section III provides some theoretical foundations regarding uncertainty, such as Fuzzy Systems Theory (FST), RFS, Grey Systems Theory (GST), and others, as well as the posited/utilized approach for contending with uncertainty; it also delineates some preliminary experimental thoughts/forays regarding the posited performance acceleration approach. Section IV concludes with a discussion, presents some preliminary reflections, puts forth envisioned future work, and the acknowledgements close the paper.

II. Background Information

Generally speaking, the concept of a fuzzy list should be fairly self-evident and intuitive; however, for the purposes of this paper, it shall be explicitly defined as being related to membership candidacy amidst temporally fluctuating dotted-line relationships, wherein the criteria for membership as well as the desire for membership may be fluid. A fuzzy list may have various types of membership, different grades of membership within the types of membership (e.g., primary, secondary, etc.), varying degrees of membership participation within the grade, and the enumeration goes on. For the most part, T2FS well captures many of the aspects for such a fuzzy list, as underscored by the fact that interval T2FS are among the most prevalent (wherein the secondary grades equate to one) approaches in contemporary times. For example, the notion of a Footprint of Uncertainty (FOU) for, say, primary memberships can be quite useful, as it can readily be depicted in two-dimensions (as contrasted to three-dimensions, which is more complex to depict and decipher) [9]; FOUs lend toward intuitiveness, which is central for explainability and justification, and is essential prior to moving to Type-3 Fuzzy Neural Networks [10]. For example, as noted in our prior work [11], a more intuitive constituent representation can better lend toward analysis; a particular example is provided, wherein "a very large matrix A" is "being factorized into, let us say, matrices B and C" and "ultimately, the desire is that all the involved matrices have no negative elements" [11][12]. However, if a standard method of matrix factorization, such as Singular Value Decomposition (SVD) is leveraged, "the resulting SVD-based lower rank representation leads to both positive and negative elements (which is the antithesis of the intent to have no negative elements), thereby making interpretation quite challenging due to the ensuing ambiguity" [11]. Yet, if the approach vector of Nonnegative Matrix Factorization (NMF) is leveraged, given "the inherent constraint that the factorized matrices be comprised of non-negative (i.e., positive) elements," it can be readily ascertained that "the involved approximation/representation as the sum of positive elements (e.g., matrices, vectors, integers) is more intuitive, logical, and naturalistic given the matrices of positive integers," which can then "facilitate a more robust interpretation of the original matrix data, as it segues to a more intuitive structural representation by parts" [11]. Thus, when this particular SOT is utilized (e.g., commencing with NMF and concluding at a Continuous Wavelet Transform (CWT), an interesting unsupervised Machine Learning (ML) pathway for HVD emerges; however, it should be noted that there are three challenges to be addressed along this sherpa-like SOT pathway: (1) mitigating against selection bias via a more balanced determination of entropy weights in the CWS (i.e., an Adaptive CWS or ACWS), (2) discerning VSNO in HDD to successfully operationalize the ICSM, and (3) determining apropos HCM for QA/QC purposes and the task at hand.

A. The Challenge of Bias in CWS

According to Chakraborty, Chen, and others, in numerous instances, CWS often utilize Subjective Methods (SMs) that "often lead to biased estimation of criteria weights," which "often results in biased results" [13][14]. SM inherent biases might include, by way of example, particular parameters/certain indices omitted/opted for, etc. A potential mitigation approach to contend with this selection bias, among others, is to use the referenced ACWS, which takes into consideration and complements the SMs with Objective Methods (OMs). The SMs and OMs comprise the constituent elements of the encompassing Multi-Attribute Decision Making (MADM) and Multi-Objective Decision Making (MODM) frameworks; in turn, the MADM and MODM comprise Multi-Criteria Decision Making (MCDM). The objective, then, is to devise a well counterpoised MADM/MODM SM/OM (MMSO) construct for deriving the requisite robust entropy weights. This mitigation against selection bias, via the proposed MMSO construct, constitutes one of the contributions of this paper.

It is essential that the MMSO packages, such as those derived from Github, be gauged for the metrics of Transparency (TY), Flexibility (F), Consistency (C), and Performance (P) (TFCP). After all, TY is vital when endeavoring to mitigate against selection bias, wherein the need is well noted by the Explainable AI (XAI) movement. F is critical for extensibility, particularly when the literature is continuously being refreshed with new hybrid approaches [15]; accordingly the chosen MMSO should be amenable for being combined with other packages. C should be axiomatic given the desire for accuracy (as well as precision, concurrently, to the degree possible). P is a driving factor, which dictates whether the considered approach is feasible or not from a computational vantage point; there are also several underlying facets shaping P, such as that of numerical stability, potential stagnation at local optima (or a differing lack of convergence), etc.

A prospective viable counterpoised construct, such as that experimented with for this paper, might be of the following composition. With regards to MADM SM, the classic Point Allocation (PA) and Saaty's Analytic Hierarchy Process (AHP) might be utilized. Weighted Aggregated Sum Product Assessment (WASPAS) can be utilized as well [16][17][18]. For MADM OM, Diakoulaki et al.'s CRiteria Importance through Intercriteria Correlation (CRITIC) and Hwang & Yoon's Technique of Order Preference by Similarity to an Ideal Solution (TOPSIS) might be selected. Fuzzy VIšekriterijumsko KOmpromisno Rangiranje (VIKOR) can be used as well [19]. For MODM SM, Brauers & Zavadskas' Multi-Objective Optimization by a Ratio Analysis plus the Full Multiplicative Form (MULTIMOORA) can be utilized. MODM OM is a slightly more complicated matter, as Hanine et al. and others have asserted that Mathematical Programming Methods (MPM), ML, and Integrated Approaches (IA) should be considered (not only for MODM OM, but also the other MMSO) [20]; accordingly, a bespoke architectural construct described in [21] is utilized. In essence, as noted in [21], it is a Constriction Factor (CF)-Particle Swarm Optimization (PSO)-Robust Convex Relaxation (RCR)-Long Short-Term Memory (LSTM)-Deep Convolutional Neural Network (DCNN) (CPRLD), which will be discussed in Section III.

For the fuzzy domain, the use of Spherical Fuzzy Sets (SFS), as introduced by Gundogdu et al. [22], are of value-added proposition as they contribute to some of the specific needs in risk assessment and decision-making (e.g., SFS can accommodate uncertainty and the notion of imprecision/quantitative exactitude for MADM problems) [23]; Ortega et al. affirm with regards to the proposed for MADM SM [24]. Kahraman et al. and Sharaf affirm with regards to that for MADM OM [25][26]. Gundogdu et al. affirm with regards to that for MODM SM [27]. Hopefully, affirmation of the MODM OM selection will occur in the not too distant future. For those cases, wherein the MMSO conjoined multiple packages, the various pairings were designed to be well counterpoised. For example, the pairing might involve complementary structures (e.g., PA is matrixed while AHP is hierarchical) and/or complementary roles (e.g., criteria weights can be derived by CRITIC and, subsequently, ranked by TOPSIS), as noted in prior works.

The notion of an ACWS has long been supported and promulgated by researchers; these include, among others, Li et al., Chai, Skondras, and others [28][29][30]. Along this vein, an ACWS is used for the preliminary experimention and operationalized by the described MMSO construct.

B. The Challenge of Discerning VSNO for ICSM

Laborde, Vitelli, and others have noted that discerning sparse solutions (e.g., subspace clusters) in HDD is a formidable feat, and certain language, such as "have not been designed for" HDD and "has not been explored prior to the writing of this paper," has been used by researchers to underscore the fairly nascent state of the research being conducted within this arena [31][32]. Suffice it to say, the process of HVD is non-trivial. Many researchers have noted (and the literature is rife with examples) the sensitivity of prototypical clustering classifiers (e.g., K-Means, K-Nearest Neighbor, etc.) to the placement of the initial seeds [33], noise [34], and the lackluster efficacy when confronted by varying cluster sizes, densities, shape, etc. (i.e., constraints of the employed metaheuristics) [35]. The efficacy of these classifiers are predominantly determined by the underlying measures utilized; exemplar measures include distance (e.g., Euclidean, Manhattan, Minkowsky, etc), similarity/dissimilarity (e.g., Cosine, Dice, Jaccard, etc.), and others. Measures for HDD need to be tailored for such; HVD needs to be gauged for its validity (e.g., goodness) and quality (e.g., robustness) of the resultant clusters.

The efficacy of the HVD is predicated upon the underlying workflow sequences, as well as the involved measures. For example, to discern VSNO in HDD, similitude needs to be gauged in HDD. Govaert and Nadif noted that insight could be gleaned from the relationships among the subspace elements, such as that of submatrices (e.g., homogeneous subsets of data). Along this vein, Wang also advocated for a subspace approach [36]. Extrapolating upon this, others (e.g., Majdara, Li, Xianting, etc.) proposed density-based approaches, and yet others (e.g., Zhao, Du, Lu, etc.) put forth grid-based approaches [37][38]. Still others have introduced hybridized approaches; for example, Agrawal et al. introduced a density-based and grid-based approach referred to as Clustering in QUEst (CLIQUE) [38]. As a follow-on enhancement to CLIQUE, Merging Adaptive Finite Intervals And (MAFIA) was introduced by Nagesh et al. [39]. Still other approaches include those that are Wavelet Transform (WT)-based. This should be of no surprise, since WT are a recognized method "to summarize high-dimensional data in a few numbers" [40].

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The literature has shown various researchers promulgating various WT in HDD. For example, Bruce et al., Qu et al, and Tuna et al. have championed Discrete WT (DWT). Interestingly, as noted in both prior work and ongoing unpublished work (e.g., "AI-Facilitated Dynamic Threshold-Tuning for a Maritime Domain Awareness Module," "AI-Facilitated Selection of the Optimal Nondominated Solution for a Serious Gaming Information Fusion Module," etc.), for the purposes of this paper, the Continuous Wavelet Transform (CWT) has certain advantages over other forms of WT [41]. For example, CWT tends to have better resolution than, say, DWTs, which are not only characterized by constrained sampling and fixed windowing, but are also beset with spectral leakage. The Stationary Wavelet Transform (SWT), as an extension of DWT, is not necessarily beset with spectral leakage, but it does tend to be overdetermined/redundant in its representation. Given that CWTs are particularly amenable to time series analysis and are very well suited for implementation aboard the CPRLD architecture, which has "successive convolutional layers (which contain the cascading of ever smaller 'CWT-like' convolutional filters)," CWTs are the preferred WT embodiment [21]. The implementation described in [11][21][42] and in Section III, along with an additional DCNN-3 to assist with complexity reduction, is another contribution of this paper.

As noted in the beginning of this Section II and in [21], an effective SOT begins with MMF, progresses to a Gaussian Composite Model (GCM), which then undergoes a transformation to a Multiresolution Matrix Factorization (MMF), a Corresponding WT (CORWT), and "an Enhanced CORWT (ECORWT), which was operationalized by way of a CWT PyWavelet Schema" (CPS) [11]. This SOT progression was operationalized by the CPRLD (and the additional DCNN-3), and the resultant CWT (seguing from the ECORWT) is utilized for fleshing out the wavelet space-based mapping ecosystem, as a facilitating precursor to the HVD operationalization.

The HVD runway is underpinned with soft clustering; this provides the requisite versatility of more granular and variegated classification; this is distinguished from hard clustering, wherein there is classification into only one cluster. The characterization of soft versus hard clustering should be reminiscent of T2FS (as contrasted to the Type-1 Fuzzy Set or T1FS, which only accommodates membership invariableness). In addition, ongoing work has indicated that, Three-Way Soft Clustering (TWSC) nicely suffices for the purposes at hand; TWSC is suitably defined via "samples in the positive region as belonging to the cluster, samples in the boundary region as partially belonging to the cluster, and samples in the negative region as not belonging to the cluster" [43].

The notion of TWSC (and hybridizations thereof) is well supported and espoused by researchers such as Wang, Yu, Ali, Yang, and others [44][45][46][47]. Also, the "fuzzy clustering" and CWT approach seems to be a well recognized combinatorial, as evidenced by the works of Jafari, Kumar, and others [48][49]. Taking these two described approaches, the HVD for this paper is operationalized by the TWSC and CWT combinatorial (aboard the CPRLD with DCNN-3). This updated implementation is another of the contributions of this paper.

C. The Challenge of Selecting Apropos HCMs

With regards to the process of HVD, Ko and others view the underlying HCM via the groupings of: (1) data certainty, and (2) data uncertainty [51]. The predominant share of HCM center upon (1), and Tavakkol et al. even noted, "To the best of our knowledge, there is not any clustering validity index in the literature that is designed for uncertain objects and can be used for validating the performance of uncertain clustering algorithms" [52]. In essence, the particular area is in an emerging state. By way of background, HCM are generally classified into three groupings: External Measures (EMs), Internal Measures (IMs), and Relative Measures (RMs). EMs are more disposed towards leveraging cluster structures/resultants from data sources not necessarily intrinsic to the clusters and data at-hand. IMs are more focused upon the affinity aspect, which exists predominantly within the clusters and data at-hand, as well as place an emphasis on compactness/cohesion. RMs, which are sometimes construed to be an extension of IMs, have a tasking that is more akin to comparing/contrasting variegated cluster structures by leveraging what is known about various EMs and IMs, etc. Exemplars of EM include F-Measure (FM), Normalized Mutual Information (NMI), etc. Examples of IM include Calinski-Harabasz (CH), Davies-Boulding (DB), etc. RM can include Dunn-Index (DI), Maulik-Bandyopadhyay (MB), etc. As noted by Vendramin and others, RM can be construed to be a subset of IM (which can be construed to encompass "Optimization-like Criteria" or OLC and "Difference-like Criteria" or DLC), and RM can refer, in particular, to DLC, wherein a baseline reference can be established and utilized to determine relative improvement(s) over a certain time frame [53][54]. Extrapolating upon other prior/concurrent work, this paper utilizes a DLC orientation when treating RM, as it more closely resonates with the notion of a baseline reference in the evaluation of HVD performance over a period of time.

Various researchers, such as Vendramin et al., have illuminated some interesting prospective performance acceleration opportunities. For example, as discussed in [54], while the seminal figures of "Milligan and Cooper" had "evaluated" the IM/RM of "McClain-Rao" (MR) as an OLC, Vendramin et al., among others, found that MR "performed signficantly better (eight times more accurately)" when transforming DLC to OLC (e.g., better results) prior to any evaluation [54]. An examination began in prior/concurrent work and continued on into the considerations of this paper. Precursor work involved determining which HCMs have been considered for facilitating the DLC Candidate List (CL). Ensuing work centered upon conducting an extensive literature review and ascertaining which HCMs were prospective candidates that warranted further exploration. The underlying logic centers upon the conviction that if the classification related to DLCs can be augmented, and the involved transformations, such as that of DLC to OLC, can be accelerated, then the ensuing enhanced HVD process can, likely, better provide for and facilitate the overall involved underpinning AI/ML processes.

The need for a robust HCM apparatus is underscored by Tavakkol, Vendramin, and others. Comparisons as well as performance acceleration opportunities have been looked to by Arbelaitz et al., Pakgohar et al. (who differ from Vendramin et al. in terms of their perspective on viable conversion methods, such as DLC to OLC), as well as many other researchers. Suffice it to say, the exploration of useful transformation methods, both from Vendramin's and Pakgohar's vantage points, are being explored in an ongoing fashion. Both have valid points, and for the purposes of this paper, Vendramin's posits were examined.

III. Theoretical Foundations & Experimentation

The ascertainment of a DFL is a non-trivial challenge that draws upon various foundational works to tackle the notion of uncertainty. There are several well recognized and acknowledged scientific method-based formalized systems for addressing uncertainty. These include, among others, the works of Cardano, Pascal, Fermat, Bernoulli, Laplace, as well as the contributions of others, to Probability [& Statistics] Systems Theory (PST), Zadeh's FST [55], Yang & Hinde's RFS (an extension of Pawlak's Rough Set (RS) and FST) (e.g., RWS that contain inconsistent data) [56], and Deng's GST [57]. This paper capitalizes upon these and others to facilitate the addressing of uncertainty.

Uncertainty is necessarily counterpoised with ambiguity, and both are impacted via the time available along a Decision Engineering Pathway (DEP); the referenced time availability can be classified as: "(1) Uncompressed Decision Cycles (UDC), and (2) Compressed Decision Cycles (CDC)" [58]; under a CDC paradigm or "tight time constraints, it accepts higher uncertainty (i.e., sparse data) given the condition of lower ambiguity" (i.e., Lower Ambiguity, Higher Uncertainty or LAHU), and this roughly translates to the consideration that an isomorphic scenario has manifested previously within the available historical data. Conversely, if there exists a condition of higher ambiguity" (i.e., Higher Ambiguity, Lower Uncertaintylo or HALU), wherein the isomorphic scenario is nonexistent within the historical data, there will be a proactive seeking of more data "to lower uncertainty" so as to move towards a more acceptable state [58]. This described LAHU/HALU Module (LHM) is key for DEP by determining whether more data is needed or not, such as by way of the resultants from the ICSM [58]. Restated, under a LAHU paradigm, the LHM will not seek further resultants, so the resultants of the ICSM may not be required.

It can be seen that the HVD underpinning the LHM is critical, and the two pillars are that of TWSC and CWT. For TWSC, the format/lexicon of Wang is utilized, "assuming that $TWSC = \{TWSC_1, \ldots, TWSC_x\}$ is a family of clusters within ecosystem $E = \{E_1, \ldots, E_y\}$, and $_m = 1, \ldots, x$, it can resemble (1):

$$TWSC_m = (PR(TWSC_m), BR(TWSC_m)) \tag{1}$$

wherein the Positive Region $(TWSC_m) = PR(TWSC_m)$, Boundary Region $(TWSC_m) = BR(TWSC_m)$, and the Negative Region $(TWSC_m) = NR(TWSC_m) = E - (PR(TWSC_m) \cup BR(TWSC_m))$, and wherein $PR(TWSC_m)$ is the set of objects that are definitively a part of $TWSC_m$, $BR(TWSC_m)$ is the set of objects that possibly belong to $TWSC_m$, and $NR(TWSC_m)$ is the set of objects that definitively are not part of $TWSC_m$." [59].

On the CWT side, The CWT is defined by (2) [50]:

$$CWT(\alpha, \beta) = \int I(t) MW(t) dt = \frac{1}{\sqrt{\alpha}} \int I(t) MW_\alpha^\gamma \left(\frac{t-\beta}{\alpha}\right) dt \tag{2}$$

wherein I is the input, α is the scale factor, β is the translation factor, γ is the complex conjugate, and $MW(t)$ is the Mother Wavelet function [50].

In contemporary times, DFLs are formulated (to the extent that they are) based upon a myriad of criteria at a particular instance in time. Yet, oftentimes, SMs (as discussed in Section IIA, are heavily used for the criteria weighting, which segues to selection bias. Hence, it is critical to buttress the TWSC-CWT for the HVD by way of the MMSO and HCM. Taking the MMSO first, the composition is shown in Figure 1, wherein the orange-outlined boxes are MODM-related and the blue-outlined boxes are MADM-related; the same holds true for the arrows associated with these colors.

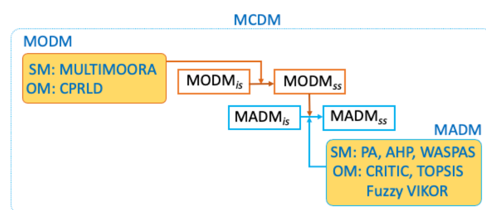

Figure 1. MCDM's $MODM_{ss}$ Facilitation of $MADM_{is}$ to $MADM_{ss}$ Progression and Experimental MMSOs

As noted in [21], the "MODM solution set ($MODM_{ss}$)" can indeed facilitate "the MADM input set ($MADM_{is}$) to $MADM_{ss}$ progress," particularly when leveraging a specific bespoke formulated notion to help operationalize the HVD, via the following lynchpin — a bifurcated Relationship Membership Stream (RMS) [21]. First, FST (HVD-1a) is considered along with RFS (HVD-2a), which is an extension of HVD-1a and RS (HVD-1b). Second, GST (HVD-2b) can improve the robustness of HVD-2a. As noted in various related prior/concurrent work leading up to this paper, if the RMS (e.g., entity, attribute, etc.) "is discontinuous," HVD-2b "can be leveraged; otherwise, "given a continuous/continuous alternative paradigm, then other" PST "approaches might be utilized, such as Information Entropy Methods (IEMs)" (HVD-3), "whose strength resides in ascertaining 'unknown attribute weights'" [60][21]. Of course, the RMS can be either discontinuous or continuous (e.g., pulsed, rather than continuous), and this is reflected in purple in Figure 2 below. The constituent parts of MCDM (i.e., MADM and MODM) are reflected in blue, and the various HVD are depicted in green.

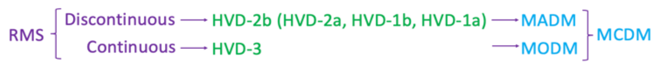

Figure 2. RMS Paradigms for the MMSO Support of HVD

The orchestration of the MMSO can be seen with the endeavored counterpoising of MADM and MODM within the MCDM construct.

Any existing selection bias (hopefully, somewhat mitigated by the MMSO construct) is also compounded by the frequent dearth of a robust Quality Assurance/Quality Control (QA/QC) mechanism. The QA/QC mechanism is very much needed to undertake apropos validity testing. Furthermore, insights into related entities (which might be highly relevant in determining membership affinity) is often lacking, which can skew the membership value pertaining to such determinations. Accordingly, various HCM experimentation approaches had been initiated in prior/concurrent work, and the exploration continues on into this paper. This is reflected in Table II below.

TABLE II. HCM FOR HVD EXPERIMENTATION FACETS FOR DLC TO OLC PROSPECTIVE CANDIDACY

I	II	III	IV	V	VI	VII	VIII	IX
MR	DLC	Min;Elbow	WB	$O(nN^2)$	✓	✓	✓	
BH	DLC	Max_{diff};Elbow	W	O(nN)				
DI	OLC	Max;Max;Max	WB	$O(nN^2)$		✓	✓	
PBM	OLC	Max;Max	WBD	$O(n(k^2+N)$				✓
TW	DLC	Max_{diff};Elbow	W	O(nN)				
PB	OLC	Max;Max	WB	$O(nN^2)$				✓

In essence, Column I lists some exemplar HCMs: MR, Ball-Hall (BH), DI, Pakhira-Bandyopadhyay-Maulik (PBM), Trace(W) (TW), and Point-Biserial (PB). Column II notes the DLC/OLC presort, as presented by Vendramin and Liu [54][61]. Column III provides some trade-off inflection points in the way of "elbows" (e.g., positive concavity) or "knees" (e.g., negative concavity) as well as the method posited to ascertain optimality (e.g., partition), wherein "the smallest index value" is denoted by Min and the largest by Max [63]; As noted in [63], Max_{diff} refers to the optimal K seguing to the maximum difference "between … successive slopes" [61][62][63]. Column IV uses Powell's convention/nomenclature of "Within-cluster (W), Between-cluster (B), and full Dataset (D)" [62]. Column V delineates the computational complexity [38][54][64]. The P of TFCP for the various normal distributions, increasing degree of overlap, global optimum, as well as paradigms that are generally affirmed (and are affirmed by other benchmarks) are checked off for the pertinent cells of Columns VI, VII, VIII, and IX, respectively, and commonalities are noted in green [38][54][64].

The various aspects discussed in this Section III are operationalized aboard the CPRLD (with DCNN-3) construct, which involves a "DCNN" and "Generative Adversarial Network (GAN)" amalgam (DCNN-GAN or DCGAN)" to avoid mode failure/collapse (i.e., "Helvetica Scenario") as well as a distinct/disparate DCNN to handle the RCR (DCNN-1) and yet another DCNN to handle the CWT derivation (DCNN-2)," via the SOT previously discussed, as the chosen approach vector for managing the HDD mappings [47][21][65]. Furthermore, a DCNN-3 is added to buttress the handling of additional non-convex problems that may be spawned from the RCRs, and this is shown in Figure 3.

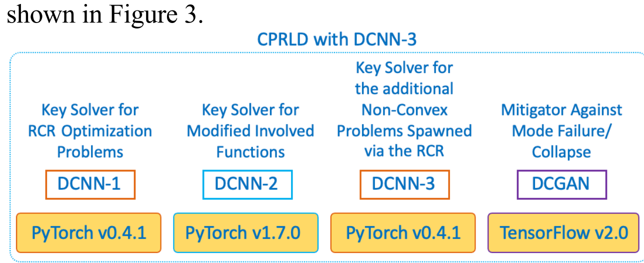

Figure 3. CPRLD Construct with additional DCNN-3

As a continuation of prior/concurrent work, this paper reviewed the findings put forth (by the researchers cited within) so as to posit conducive conditions for prospective performance accelerations (e.g., DLC candidates); as also noted in concurrent work, and to re-articulate the point, it turns out that MR, BH, and TW potentially constituted viable DLC candidates, among others, while DI, PB, and PBM exhibited performance possibilities, which need to be explored as part of future work. Some of categorical commonalities of this prospective candidate set are green highlighted. In addition, the disparities (and consensus) are also noted by way of the the varying colors. To re-articulate a point from prior/concurrent work, as pertains to the DLC commonality (e.g., for MR), the "value that *minimizes* the index indicates the optimal cluster number," while for BH, the value that *maximizes* the "difference between levels is used to indicate the optimal solution." It should also be noted that — unlike stopping rules, which constitute a cessation of the iterations — the DLC paradigm has the value-added proposition of exhibiting peaks (e.g., Max, Min) at OLC areas [66]. In essence, the DLC to OLC prospective candidacy determination constituted potential performance acceleration opportunities. In addition, central to the determination of appropriateness, for the various HCMs, was the evaluation for prospective performance acceleration via the previously cited SOT (i.e., NMF->GCM->MMF->CORWT->ECORWT->CPS) prior to any assessment.

The experimentation herein operationalized the discussed framework of this paper, as shown in Figure 4. The novelty of the framework was rooted by several contributions, among others: (1) the mitigation against selection bias, via the proposed construct, which included the additional use of WASPAS and Fuzzy VIKOR, (2) The specific implementation, via the enhanced CPRLD architecture, which capitalized upon the benefit of using a DCNN-3, (3) the TWSC and CWT combinatorial (aboard the CPRLD with DCNN-3) to operationalize the HVD, and the utilization of the LHM to optimize performance (e.g., determining that it was not necessary to seek further resultants from the ICSM) along the DEP.

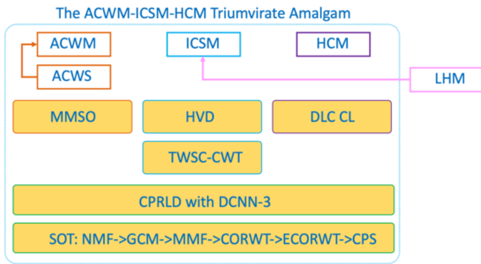

Figure 4. The ACWM-ICSM-HCM Triumvirate Amalgam with LHM

The Section III experimentation section explored a triumvirate amalgam of (1) apropos HCM that are data uncertainty-centric, (2) an ICSM to gauge similitude, via the discernment of VSNO clusters in HDD (a.k.a., HVD), and (3) an Adaptive Criteria Weighting Methodology (ACWM), underpinned by an ACWS, to derive the requisite entropy weights. Collectively, these three separate modules, of the described triumvirate approach, demonstrated the viability of the illuminated DFL pathway, which exhibited some promise during the preliminary experimentation phase.

IV. Conclusion & Future Work

The nexus of AI with the notions of uncertainty and ambiguity represents a complex environment, wherein the interpretation may be fluid amidst various computational complexities. For example, AI/ML rule-based systems do not well handle ambiguity. Even for the more adaptable AI/ML systems, uncertainty besets various aspects, such as data (e.g., incomplete, noisy), perception (e.g., the inherent limitations of the sensors), models (e.g., hyperparameters to be tuned, optimization algorithms/heuristics utilized), algorithms (e.g., different NMs segue to different results for the identical problem), etc. This constitutes a herculean challenge, as AI for RWS must necessarily not only provide posits, but it necessarily needs to provide a level of certainty regarding those posits. This gauging of uncertainty in RWS applications (e.g., medical diagnoses, system-initiated emergency braking for vehicles, etc. [67]) directly impacts reliability and safety. This is especially the case for semi-autonomous and autonomous RWS. Hence, the efficacy of the LHM and discussed triumvirate (and the crucial DCNN-3 component) are central to the discussion.

With regards to the triumvirate, the formulation of a robust DFL is indeed a non-trivial feat. Among various potential approaches, the pathway explored within this paper was that of a specific triumvirate amalgam: ACWM-ICSM-HCM. For the preliminary experimentation touched upon herein, the underpinning ACWS of the ACWM was operationalized by the delineated MMSO construct, the HVD was effectuated the TWSC-CWT, and the selection of apropos HCMs was explored for potential performance enhancement opportunities; in particular, the prior works of well-known researchers were examined and, in many cases, affirmed. Suffice it to say, the ACWM-ICSM-HCM triumvirate constitutes only the beginnings of a prospective DFL pathway, and further exploration and experimentation, such as that referenced in Section III, regarding future work, remains to be performed. Having said that, the SOTs described herein, and their potentialities, seem to warrant further investigation. Likewise, the MMSO Construct shown in Figure 1 seems to be a viable foundation upon which to build; similarly, the RMS shown in Figure 2 has well captured many of the foundational contributions from the literature and seems to remain a viable formulation. Finally, the Table II excerpt illuminates a sampling of the various categorical parameters that need to be further explored in future work to further flesh out the posits of Vendramin, Pakgohar, and others. Overall, the presented triumvirate approach did indeed endeavor to address the selection bias issue, via the semblance of an ACWM for a more balanced criteria weighting schema. It also incorporated the thoughts of seminal figures and contemporary researchers with regards to discernment of VSNO in HDD. It further considered HCM in the context of practicality: QA/QC and the task at hand. For these reasons, the ACWM-ICSM-HCM triumvirate approach, as a contribution to the DFL challenge (and the overarching complex problem of risk-prediction-decision in advanced AI/ML systems), warrants further investigation, as it covers a thematic that is crucial for IF in dynamic RWS, and the paper reviewed a series of challenges for more optimal MCDM and DEP. The proposed logic/premise (the notion that if the classification related to DLCs can be enhanced, and the involved transformations, such as that of DLC to OLC, can be expedited, then the ensuing facilitated HVD process can likely better support the overall involved underpinning AI/ML processes) exhibits promise. Moreover, the proposed triumvirate approach seems to have potential in addressing the stated overarching objectives.

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